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ψ -Adamar ma’nosidagi kasr tartibli integral va hosilalar

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Annotatsiya: So‘nggi yillarda kasr tartibli analiz matematik tadqiqotlarning muhim yo‘nalishlaridan biriga aylandi. Kasr tartibli integrallar va hosilalar ko‘plab amaliy va nazariy masalalarni yechish uchun samarali matematik apparat hisoblanadi. Ular fizika, biologiya, suyuqliklar mexanikasi, boshqarish nazariyasi, signallarni qayta ishlash va turli murakkab jarayonlarni matematik modellashtirishda keng qo‘llaniladi. Ushbu maqolada kasr tartibli integral va Adamar ma’nosidagi hosila tushunchalari ko‘rib chiqiladi. Kesmada aniqlangan kasr tartibli integrallar va Adamar tipidagi hosilalarning ta’riflari keltiriladi va ularning asosiy xossalari tahlil qilinadi. Shuningdek, Lebeg fazosida aniqlangan operatorlarning chegaralanganligi, Riman-Liuvill va Adamar tipidagi kasr tartibli integrallar orasidagi bog‘lanish hamda ba’zi muhim teorema va natijalar qaralgan.

Kalit so‘zlar: Kasr tartibli hosila, kasr tartibli integral, Adamar integrali, Riman–Liuvill integrali, Lebeg fazosi, differensial operator, matematik modellashtirish, kasr analiz.

Аннотация: В последние годы анализ дробного порядка стал одним из важных направлений математических исследований. Интегралы и производные дробного порядка являются эффективным математическим аппаратом для решения многих практических и теоретических задач. Они широко применяются в физике, биологии, механике жидкостей, теории управления, обработке сигналов и математическом моделировании различных сложных процессов. В данной статье рассматриваются понятия дробного интеграла и производной в смысле Адамара. Приводятся определения дробных интегралов и производных типа Адамара, определенных на отрезке, и анализируются их основные свойства. Также рассмотрены ограниченность операторов, определенных в пространстве Лебега, связь между интегралами дробного порядка типа Римана-Лиувилля и Адамара, а также некоторые важные теоремы и результаты.

Ключевые слова: Производная дробного порядка, интеграл дробного порядка, интеграл Адамара, интеграл Римана-Лиувилля, пространство Лебега, дифференциальный оператор, математическое моделирование, дробный анализ.

Abstract: In recent years, fractional analysis has become one of the important areas of mathematical research. Fractional integrals and derivatives are an effective mathematical apparatus for solving many practical and theoretical problems. They are widely used in physics, biology, fluid mechanics, control theory, signal processing, and mathematical modeling of various complex processes. This article examines the concepts of the fractional integral and the derivative in the Adamarian sense. Definitions of fractional integrals defined on a segment and derivatives of the Adamard type are given, and their main properties are analyzed. It also considers the boundedness of operators defined in Lebesgue space, the relationship between Riemann-Liouville and Adamar fractional integrals, as well as some important theorems and results.

Keywords: Fractional derivative, fractional integral, Adamar integral, Riemann-Liouville integral, Lebesgue space, differential operator, mathematical modeling, fractional analysis.

Kirish: Zamonaviy matematikaning muhim yo‘nalishlaridan biri bo‘lgan kasrlar hisobi (fractional calculus) differensial va integral operatorlarni butun tartibdan tashqarida umumlashtirishga asoslangan. Kasr hosilalar va integrallar nazariyasi XVIII asrda Leybnis tomonidan ilgari surilgan g‘oyalarga borib taqaladi. Keyinchalik bu soha ko‘plab olimlar tomonidan rivojlantirildi va bugungi kunda matematika va tabiiy fanlarning turli sohalar`ida keng qo‘llanilmoqda.

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Kasr tartibli analizning muhim yo‘nalishlaridan biri Riman-Liuvill, Kaputo va Adamar tipidagi kasr hosilalar nazariyasidir. Bu operatorlar turli fizik va texnik jarayonlarni tavsiflashda, ayniqsa, xotira effektiga ega tizimlarni modellashtirishda muhim ahamiyatga ega. Shuningdek, chiziqli bo‘lmagan differensial tenglamalar, integral tenglamalar va matematik fizika masalalarini tadqiq qilishda keng qo‘llaniladi.

Kasr tartibli integrallar va Adamar tipidagi hosilalar logarifmik yadrolarga asoslangan va ko‘plab muhim xossalarga ega. Bunday operatorlar yordamida murakkab matematik modellarni aniqroq ifodalash mumkin. Shuning uchun Adamar tipidagi kasr tartibli operatorlarni o‘rganish zamonaviy matematik analizning dolzarb masalalaridan biri hisoblanadi.

Ushbu ishda kesmada aniqlangan kasr tartibli integrallar va Adamar tipidagi hosilalar o‘rganiladi va ularning asosiy xossalari tahlil qilinadi. Shuningdek, bu operatorlarning Lebeg fazosidagi chegaralanishlari, ba’zi teoremalar va ularning natijalari keltirilgan.

Asosiy qism:

1-§. Kesmadagi ψ -Adamar tipidagi kasr integral va hosila

1.1-ta’rif Aytaylik $\alpha>0,\ -\infty\le a<b\le\infty,\ \mu\in\mathbf{C},\ \varphi$ funksiya $[a,b]$ kesmada integrallanuvchi va $\psi\in C^1([a,b])$ bo‘lib, musbat o‘suvchi funksiya hamda barcha $x\in(a,b)$ lar uchun $\psi'(x)\neq 0$ bo‘lsin. U holda $[a,b]$kesmada boshqa ψ funksiyaga nisbatan α tartibli va μ parametrli ψ -Adamar tipidagi chap va o‘ng tomonli kasrli integrallar mos ravishda quyidagicha aniqlanadi:

$$\left(J^{\alpha,\psi}_{a+,\mu}\varphi\right)(x)=\frac{1}{\Gamma(\alpha)}\int_a^x\left(\frac{\psi(t)}{\psi(x)}\right)^{\mu}\left(\ln\frac{\psi(x)}{\psi(t)}\right)^{\alpha-1}\varphi(t)\frac{\psi'(t)}{\psi(t)}dt,\ x>a, \qquad (1.1)$$

va

$$\left(J^{\alpha,\psi}_{b-,\mu}\varphi\right)(x)=\frac{1}{\Gamma(\alpha)}\int_x^b\left(\frac{\psi(x)}{\psi(t)}\right)^{\mu}\left(\ln\frac{\psi(t)}{\psi(x)}\right)^{\alpha-1}\varphi(t)\frac{\psi'(t)}{\psi(t)}dt,\ x<b. \qquad (1.2)$$

1.2-ta’rif. Aytaylik $\alpha>0, n-1<\alpha\le n, n\in\mathbf{N},\ -\infty\le a<b\le\infty,\ \mu\in\mathbf{C},\ f$ funksiya $[a,b]$kesmada integrallanuvchi va $\psi\in C^1([a,b])$ bo‘lib, musbat o‘suvchi funksiya hamda barcha $x\in(a,b)$ lar uchun $\psi'(x)\neq 0$ bo‘lsin. U holda $[a,b]$ kesmada boshqa ψ funksiyaga nisbatan α tartibli va μ parametrli ψ - Adamar tipidagi chap va o‘ng tomonli kasrli hosilalar mos ravishda quyidagicha aniqlanadi:

$$\left(D^{\alpha,\psi}_{a+,\mu}f(x)\right)=\left(\psi(x)\right)^{-\mu}\left(\frac{\psi(x)}{\psi'(x)}\frac{d}{dx}\right)^n\psi^{\mu}(x)J^{n-\alpha,\psi}_{a+,\mu}f(x)=$$

$$=\psi(x)^{-\mu}\left(\frac{\psi(x)}{\psi'(x)}\frac{d}{dx}\right)^n\frac{1}{\Gamma(n-\alpha)}\int_a^x\left(\psi(t)\right)^{\mu}\left(\ln\frac{\psi(x)}{\psi(t)}\right)^{n-\alpha-1}f(t)\frac{\psi'(t)}{\psi(t)}dt, x>a$$

va

$$\left(D^{\alpha,\psi}_{b-,\mu}f\right)(x)=\left(\psi(x)\right)^{\mu}\left(-\frac{\psi(x)}{\psi'(x)}\frac{d}{dx}\right)^n\psi^{-\mu}(x)J^{n-\alpha,\psi}_{b-,\mu}f(x)=$$

$$=\psi^{\mu}(x)\left(-\frac{\psi(x)}{\psi'(x)}\frac{d}{dx}\right)^n\frac{1}{\Gamma(1-\alpha)}\int_x^b\left(\psi(x)\right)^{-\mu}\left(\ln\frac{\psi(t)}{\psi(x)}\right)^{n-a-1}f(t)\frac{\psi'(t)}{\psi(t)}dt, x<b,$$

bu yerda $n=[a]+1.$

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$0<\alpha<1$ uchun ψ *-Adamar ma'nosidagi o'ng va chap tomonli* kasr tartibli hosilalar formulalarni quyidagi

$$\left(D_{a+,\mu}^{\alpha,\psi} f\right)(x)=\frac{\psi^{1-\mu}(x)}{\psi^{'(x)}}\frac{d}{dx}\frac{1}{\Gamma(1-\alpha)}\int_a^x \psi^{\mu}(t)\left(\ln\frac{\psi(x)}{\psi(t)}\right)^{-\alpha} f(t)\frac{\psi'(t)}{\psi(t)}dt, x>a \quad (1.2)$$

va

$$\left(D_{b-,\mu}^{\alpha,\psi} f(x)\right)=-\frac{\psi^{1+\mu}(x)}{\psi^{'(x)}}\frac{d}{dx}\frac{1}{\Gamma(1-\alpha)}\int_a^x \psi^{-\mu}(t)\left(\ln\frac{\psi(t)}{\psi(x)}\right)^{-\alpha} f(t)\frac{\psi'(t)}{\psi(t)}dt, x<b$$

ko'rinishlarda yozish mumkin.

1.3-ta'rif. $-\infty\le a<b\le\infty$, $c\in\mathbf{R}$ bo'lib, $1\le p\le\infty$, $L_{\psi,c}^{p}(a,b)$ fazo $[a,b]$ da aniqlangan shunday Lebeg kompleks qiymatli funksiyalarning fazosi bo'lsin, ya'ni $f:[a,b]\to C$ uchun $\|f\|_{L_{\psi,c}^{p}(a,b)}<\infty$, bu yerda

$$\|f\|_{L_{\psi,c}^{p}}=\|f:L_{\psi,c}^{p}\|=\left(\int_a^b\left|\{\psi(x)\}^{c} f(x)\right|^{p}\frac{\psi'(t)}{\psi(t)}dt\right)^{1/p} \quad (1.3)$$

bo'lganda $1\le p\le\infty, c\in R$ va

$$\left\|f_{L_{\psi,c}^{\infty}}\right\|=\|f:L_{\psi,c}^{\infty}\|=\operatorname*{esssup}_{a\le x\le b}\left|\{\psi(x)\}^{c} f(x)\right|$$

bo'lganda $p=\infty, c\in R$.

Agar $c=\frac{1}{p}$ va $\psi(x)=x$ deb qaraydigan bo'lsak, u holda $L_{\psi,c}^{p}(a,b)$ fazo $L_{p}(a,b)$ fazo bilan ustma-ust tushadi va uning normasi

$$\|f\|_{L_{\rho}(a,b)}=\left(\int_a^b|f(x)|^{p}dx\right)^{1/p} \quad (1\le p<\infty)$$

va

$$\|f\|_{L_{\infty}}=\operatorname*{ess\,sup}_{a\le x\le b}|f|.$$

ko'rinishlarda aniqlanadi.

1.1-teorema. Ataylik $\alpha>0$, $1\le p\le\infty$, $0<a<b<\infty$, $\mu\in\mathbf{R}$ va $c\in\mathbf{R}$ hamda ψ musbat o'suvchi funksiya bo'lsin. Agar $\mu>c$ bo'lsa, u holda $J_{a+,\mu}^{\alpha,\psi}$ operator $L_{\psi,c}^{p}(a,b)$ fazoda chegaralangan bo'ladi, ya'ni

$$\left\|J_{a+\mu}^{\alpha,\psi}\varphi;L_{\psi,c}^{p}\right\|\le d\left\|\varphi;L_{\psi,c}^{p}\right\|,$$

bu yerda

$$d=\begin{cases}\dfrac{1}{\Gamma(\alpha+1)}\left(\ln\dfrac{\psi(b)}{\psi(a)}\right)^{\alpha}, & \mu=c \text{ bo'lganda;}\\ \dfrac{1}{\Gamma(\alpha)}(\mu-c)^{-\alpha}\gamma\left(\alpha,(\mu-c)\ln\dfrac{\psi(b)}{\psi(a)}\right), & \mu>c \text{ bo'lganda}\end{cases}$$

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1.4-ta'rif. $AC^n_{\delta^{\psi},\mu}([a,b])$ orqali, bunda $n\in\mathbf{N}$, $\delta = x\dfrac{d}{dx}$ va $\mu\in\mathbf{R}$, bo'lganda funksiyalar sinfini belgilaymiz, qaysiki $\{\Psi(x)\}^{\mu} f(x)$ funksiya $[a,b]$

$$\delta^{n-1;\psi} = \left(\frac{\psi(x)}{\psi'(x)}\frac{d}{dx}\right)^{n-1}$$

$n-1$ tartibgacha δ-hosilaga ega, va bu yeda $[a,b]$ absolyut uzluksiz:

$$AC^n_{\delta^{\psi},\mu}[a,b] := \left\{f:[a,b]\to C : \delta^{n-1;\psi}\left\{\{\Psi(x)\}^{\mu} f(x)\right\}\in AC[a,b]\right\}$$

Bunda $AC[a,b]$ dagi absolyut uzluksiz funksiyalar to'plami, qaysiki Lebeg bo'yicha boshlang'ich o'lchamli funksiyalari bilan mos tushuvchi:

$$f(x)\in AC[a,b] \Leftrightarrow f(x) = c + \int_a^x \psi(t)dt,\ \psi(t)\in L_1(a,b)$$

Quyidagi teorema o'rinli.

1.1-teorema. $\alpha>0$, $n=[\alpha]+1$, $\mu\in\mathbf{R}$ bo'lsin, ψ o'suvchi (a,b) da musbat funksiya va $f\in AC^n_{\delta^{\psi},\mu}[a,b]$. U holda ψ-Adamar tipidagi kasr hosilasi $D^{\alpha,\psi}_{a+,\mu}f$ $[a,b]$ da ψ ga nisbatan deyarli barcha yerda mavjud va uni quyidagi ko'rinishga keltirish mumkin:

$$D^{\alpha,\psi}_{a+,\mu}f(x) = \{\Psi(x)\}^{-\mu}\left\{\int_a^x \left(\ln\frac{\psi(x)}{\psi(t)}\right)^{n-\alpha-1}\frac{f'_{n-1}(t)}{\Gamma(n-\alpha)}dt + \sum_{k=0}^{n-1}\frac{f_k(a)}{\Gamma(n-\alpha+1)}\left(\ln\frac{\psi(x)}{\psi(a)}\right)^{k-\alpha}\right\},$$

bu yerda: $f_k(x) = \left(\dfrac{\psi(x)}{\psi'(x)}\dfrac{d}{dx}\right)^k [\psi^{\mu}(x) f(x)]$, $k=\overline{1,n-1}$.

1.1-natija. $0<\alpha<1$, $\mu\in\mathbf{R}$ bo'lsin, ψ $[a,b]$ da o'suvchi musbat funksiya va $f\in AC_{\delta^{\psi},\mu}[a,b]$. U holda ψ-Adamar tipidagi kasr hosilasi $D^{\alpha,\psi}_{a+,\mu}f$ $[a,b]$ kesmada ψ ga nisbatan deyarli barcha yerda mavjud va quyidagi ko'rinishga keltirish mumkin:

$$\left(\mathrm{D}^{\alpha,\psi}_{a+,\mu}f\right)(x) = \frac{f(a)}{\Gamma(1-\alpha)}\left(\frac{\psi(a)}{\psi(x)}\right)^{\mu}\left(\ln\frac{\psi(x)}{\psi(a)}\right)^{-\alpha} +$$

$$+\frac{(\psi(x))^{-\mu}}{\Gamma(1-\alpha)}\int_a^x\left(\ln\frac{\psi(x)}{\psi(t)}\right)^{-\alpha}\left((\psi(t))^{\mu} f(t)\right)' dt \qquad (1.4)$$

Xulosa: Maqolada chegaralanmagan sohada Koshi masalasining qo'yilishi va bu masala uchun Karleman funksiyasi va uning baholari keltirilgan.

R^2 ikki o'lchovli Evklid fazosida Yarmuxamedov yadrosining xossalaridan foydalanib, olingan Karleman funksiyasini integral tasviridan, ya'ni ma'lum bir sinfdagi o'suvchi bigarmonik funksiyalar sinfidagi chegaralanmagan sohada olingan Grin integral formulasidan foydalandik. Bu fazoda aniq ko'rinishi olingan yadrodan foydalanib yuqoridan baholashlarga ega bo'ldik. Olinadigan integral tasvir yordamida sohaning har bir ichki nuqtasidagi baholashlardan foydalanish juda katta ahamiyatga ega, chunki funksiyaning chegarasidagi xossalari va ichki nuqtalardagi bog'lanishlar, yagonalik teoremalari , Fragmen-Lindelef tipidagi teoremalar olishga ega bo'lamiz.

R^3 uch o'lchovli Evklid fazosida ham ma'lum bir sohalar uchun tanlab olingan Yarmuxamedov yadrosidan foydalanib yuqoridan baholashlarni oldik. Bu tasvirlar yordamida ba'zi bir shartlar asosida ma'lum bir sinflar uchun yagonalik teoremalariga ega bo'ldik.

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