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A GENERALIZED (G'/G) - EXPANSION METHOD FOR THE LOADED MODIFIED BURGERS–KDV EQUATION

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Abstract: In this article, a generalized $\left(G'/G\right)$ -expansion method is used to find the exact travelling wave solutions of the loaded modified Burgers-KdV equation. As a result, hyperbolic, trigonometric, and rational function solutions with parameters are obtained. When these parameters are taking special values, the solitary wave solutions are derived from the hyperbolic function. It is shown that the proposed method is direct and effective method, which can be applied to many other nonlinear evolution equations in mathematical physics.

Key words: Nonlinear evolution equations; generalized $\left(G'/G\right)$ -expansion method; hyperbolic solution; trigonometric solution; rational solution; loaded modified Burgers-KdV equation.

Introduction

In this paper, the solutions of the loaded nonlinear equation, one of the most significant integrally nonlinear differential equations, are explored. In the literature [1,2,3,4,5], loaded differential equations are typically called equations containing in the coefficients or in the right-hand side any functionals of the solution, in particular the values of the solution or its derivatives on manifolds of lower dimension. This type of equations were explored in works of N.N. Nazarov and N.N. Kochin. However, they did not use the term “loaded equation”. At first, the term has been used in works of A.M. Nakhushev, where the most general definition of a loaded equation is given and various loaded equations are classified in detail, for instance, loaded differential, integral, integro-differential, functional equations etc., and numerous applications are described.

Nowadays, various methods exist to solve nonlinear differential equations. For instance, Hirota direct method [6], the inverse scattering problem for the Dirac operator that studied in the works of V. E. Zakharov, A. B. Shabat [7], I. S. Frolov [8], L. A. Takhtajyan, L. D. Faddeev [9] and M. Wadati [10] and the binary Darboux transformations [11,12]. Alternatively, the (G'/G) - expansion method [13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28] is also effective in finding traveling wave solutions of nonlinear evolution equations.

In this article, the solutions of the loaded nonlinear equation are studied by usage of (G'/G) - expansion method.

Let’s consider the following loaded modified Burgers–KdV equation

$$u_t + pu^2u_x + qu_{xx} - ru_{xxx} + \gamma(t)u(0,t)u_x = 0, \qquad (1)$$

where $q(x,t)$ is an unknown function, $x \in R$, $t \geq 0$, $f(t)$ - is the given real continuous function.

Description of the generalized (G'/G)-expansion method

Let us be given a nonlinear partial differential equation in the form below

$$F(u, u_t, u_x, u_{tt}, u_{xx}, u_{xt}, ...) = 0 \qquad (2)$$

with two independent variables x and t $u = u(x,t)$ is a unknown function, F is a polynomial in q and its partial derivatives in which the highest order derivatives and nonlinear terms are involved. Now we give the main steps of the (G'/G)-expansion method [23] :

Step 1. We look for the u in the travelling form:

$$u(x,t)=u(\xi),\ \xi=x-\Omega(t), \tag{3}$$

where is parameter and $\Omega(t)$ is a continuous function dependent on t We reduce equation (2) to the following nonlinear ordinary differential equation:

$$P(u,u',u'',u''',...)=0, \tag{4}$$

where P is a polynomial of $u(\xi)$ and its all derivatives $u'=du(\xi)/d\xi$, $u''=d^2u(\xi)/d\xi^2$.

Step 2. We assume that the solution of equation (4) has the form:

$$u(\xi)=\sum_{j=0}^{m}a_j\left(\frac{G'}{G}\right)^j, \tag{5}$$

where $G=G(\xi)$ satisfies the following second order ordinary differential equation

$$G''+\lambda G'+\mu G=0, \tag{6}$$

where $G'=dG(\xi)/d\xi$, $G''=d^2G(\xi)/d\xi^2$ and λ, μ, a_j $(j=1,2,...,m)$ are constants that can be determined later, provided $a_m\neq 0$.

Step 3. We determine the integer number m by balancing the nonlinear terms of the highest order and the partial product of the highest order of (4).

Step 4. Substitute (5) along with (6) into (4) and collect all terms with the same order of $\left(\frac{G'(\xi)}{G(\xi)}\right)$, the left-hand side of (4) is converted into a polynomial in $\left(\frac{G'(\xi)}{G(\xi)}\right)$. Then, set each coefficient of this polynomial to zero to derive a set of over-determined partial differential equations for a_j $(j=1,2,...,m)$ and ξ.

Step 5. Substituting the values a_j $(j=1,2,...,m)$ and ξ as well as the solutions of equation (6) into (5) we have the exact solutions of equation (2).

Exact Solutions of loaded nonlinear equation

In this section, we will show how to find the exact solution of the loaded nonlinear equation using the (G'/G)-expansion method. For doing this, we perform the steps above for equation (1). The travelling wave variable below

$$u(x,t)=u(\xi),\ \xi=x-\Omega(t), \tag{7}$$

permits us converting equation (1) into an ordinary differential equation for $u=u(\xi)$

$$-\Omega_t'(t)u'+pu^2u'+qu''-ru'''+\gamma(t)u(0,t)u'=0, \tag{8}$$

integrating it with respect to ξ once yields

$$C-\Omega_t'(t)u+\frac{p}{3}u^3+qu'-ru''+\gamma(t)u(0,t)u=0, \tag{9}$$

where C is an integration constant that can be determined later.

We express the solution of equation (9) in the form of a polynomial in (G'/G) below

$$u(\xi)=\sum_{j=0}^{m}a_j\left(\frac{G'}{G}\right)^j, \tag{10}$$

where $G=G(\xi)$ satisfies the second order ordinary differential equation in the form

$$G''+\lambda G'+\mu G=0. \tag{11}$$

Using (10) and (11), u' and u'' are easily derived to

$$u'(\xi) = ma_m\left(\frac{G'}{G}\right)^{(m+1)} + \dots , \tag{12}$$

$$u''(\xi) = m(m+1)a_m\left(\frac{G'}{G}\right)^{m+2} + \dots . \tag{13}$$

Considering the homogeneous balance between $u'(\xi)$ and $u''(\xi)$ in equation (9), based on (12) and (13) we required that $m=1$. Taking into account considerations, the form of q is as following

$$u(\xi) = a_1\left(\frac{G'}{G}\right) + a_0, \tag{14}$$

then we know the exact view of $u'(\xi)$

$$u'(\xi) = a_1\left(\frac{G'}{G}\right)^2 + \lambda a_1\left(\frac{G'}{G}\right) + a_1\mu \tag{15}$$

using (14) and (11) $u''(\xi)$ is easily derived to

$$u''(\xi) = 2a_1\left(\frac{G'}{G}\right)^3 + 3a_1\lambda\left(\frac{G'}{G}\right)^2 + (a_1\lambda^2 + 2a_1\mu)\left(\frac{G'}{G}\right) + a_1\lambda\mu \tag{16}$$

By substituting (14)-(16) into equation (9) and collecting all terms with the same power of (G'/G), the left-hand side of equation (9) is converted into another polynomial in (G'/G).

$$\left(\frac{p}{3}a_1^3 - 2ra_1\right)\left(\frac{G'}{G}\right)^3 + (pa_1^2a_0 - qa_1 - 3ra_1\lambda)\left(\frac{G'}{G}\right)^2 +$$

$$+(-a_1\Omega_t'(t) + pa_0^2a_1 - qa_1\lambda - r(a_1\lambda^2 + 2a_1\mu) + a_1\gamma(t)u(0,t))\left(\frac{G'}{G}\right)^1 +$$

$$+(C - a_0\Omega_t'(t) + \frac{p}{3}a_0^3 - qa_1\mu - ra_1\lambda\mu + a_0\gamma(t)u(0,t))\left(\frac{G'}{G}\right)^0 = 0 \tag{17}$$

Equating each coefficient of expression (17) to zero, yields a set of simultaneous equations for a_1, $\Omega(t)$ and C as following:

$\left(\frac{G'}{G}\right)^3$: $\frac{p}{3}a_1^3 - 2ra_1 = 0$,

$\left(\frac{G'}{G}\right)^2$: $pa_1^2a_0 - qa_1 - 3ra_1\lambda = 0$,

$\left(\frac{G'}{G}\right)^1$: $-a_1\Omega_t'(t) + pa_0^2a_1 - qa_1\lambda - r(a_1\lambda^2 + 2a_1\mu) + a_1\gamma(t)u(0,t) = 0$,

$\left(\frac{G'}{G}\right)^0$: $C - a_0\Omega_t'(t) + \frac{p}{3}a_0^3 - qa_1\mu - ra_1\lambda\mu + a_0\gamma(t)u(0,t) = 0$.

By solving these the equations, we obtain the followings

$$a_1 = \pm\sqrt{\frac{6r}{p}}, \quad C = 0, \quad a_0 = \pm\frac{q + 3r\lambda}{\sqrt{6rp}}$$

$$\Omega(t) = \frac{q + 3r\lambda}{\sqrt{6rp}}t - q\lambda t - r(\lambda^2 + 2\mu)t + \int_0^t \gamma(\tau)u(0,\tau)d\tau + \Omega^0, \tag{18}$$

λ, μ, and Ω^0 are arbitrary constants.

Using (18), expression (14) can be rewritten as

$$u(\xi)=\sqrt{\frac{6r}{p}}\left(\frac{G'}{G}\right)+\frac{q+3r\lambda}{\sqrt{6rp}}, \tag{19}$$

where $\xi=x-\frac{q+3r\lambda}{\sqrt{6rp}}t+q\lambda t+r(\lambda^2-4\mu)t+6r\mu t-\int_0^t\gamma(\tau)u(0,\tau)d\tau-\Omega^0$ .The function (19) is a solution of equation (9), provided that the integration constant C in equation (9) is taken as that in (18). Substituting the general solutions of equation (11) into (19), we have three types of travelling wave solutions of the loaded nonlinear equation (1) as follows:

When $(\lambda^2-4\mu)>0$,

$$u(\xi)=\pm\sqrt{\frac{r}{6p}}\left(\frac{q}{r}+\left|\frac{q}{r}\right|\left(\frac{c_1\sinh\frac{1}{6}\left|\frac{q}{r}\right|\xi+c_2\cosh\frac{1}{6}\left|\frac{q}{r}\right|\xi}{c_1\cosh\frac{1}{6}\left|\frac{q}{r}\right|\xi+c_2\sinh\frac{1}{6}\left|\frac{q}{r}\right|\xi}\right)\right), \tag{20}$$

where $\xi=x-\frac{q+3r\lambda}{\sqrt{6rp}}t+q\lambda t+r(\lambda^2-4\mu)t+6r\mu t-\int_0^t\gamma(\tau)u(0,\tau)d\tau-\Omega^0$, c_1, c_2 and Ω^0 are arbitrary constants. It is obvious that the function $u(0,t)$ can be easily found based on expression (20).

For example, let $\gamma(t)$ is given as below

$$\gamma(t)=\left(-\sum_{j=1}^{n}j\alpha_j t^{j-1}-\frac{q+3r\lambda}{\sqrt{6rp}}+q\lambda+r(\lambda^2+2\mu)\right)\frac{1}{-\lambda\pm\sqrt{\lambda^2-4\mu}+\sqrt{\lambda^2-4\mu}\left(\frac{\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}(\sum_{j=1}^{n}\alpha_j t^j)}{\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}(\sum_{j=1}^{n}\alpha_j t^j)}\right)},$$

where $\alpha_j\,(j=1,2,...,n)$ are constants, if $c_1\neq 0$, $c_2=0$ and $(\lambda^2-4\mu)>0$,then $q(x,t)$ becomes

$$q(x,t)=-\lambda\pm\sqrt{\lambda^2-4\mu}-\sqrt{\lambda^2-4\mu}\left(\frac{\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}(x-\sum_{j=1}^{n}\alpha_j t^j)}{\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}(x-\sum_{j=1}^{n}\alpha_j t^j)}\right) \tag{21}$$

The function (21) is the solution of the following loaded modified Burgers–KdV equation.

$$q_t+qq_x-q_{xx}+\left(\sum_{j=1}^{n}j\alpha_j t^{j-1}-\sqrt{\lambda^2-4\mu}\right)q_x=0.$$

When $(\lambda^2-4\mu)<0$,

$$q(\xi)=-\lambda\pm\sqrt{4\mu-\lambda^2}-\sqrt{4\mu-\lambda^2}\left(\frac{c_1\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi+c_2\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{c_1\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi+c_2\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}\right), \tag{22}$$

where $\xi=x\pm\sqrt{\lambda^2-4\mu}t-\int_0^t\gamma(\tau)q(0,\tau)d\tau-\Omega^0$, c_1, c_2, and Ω^0 are arbitrary constants. It is clear that it is not difficult for us to find $q(0,t)$ based on expression (22). Let $\gamma(t)$ is given as below

$$\gamma(t)=\left(\sum_{j=1}^{n} j\alpha_j t^{j-1}-\sqrt{4\mu-\lambda^2}\right)\frac{1}{-\lambda\pm\sqrt{4\mu-\lambda^2}+\sqrt{4\mu-\lambda^2}\left(tg\frac{1}{2}\sqrt{4\mu-\lambda^2}\sum_{j=1}^{n}\alpha_j t^j\right)},$$

where $\alpha_j\,(j=1,2,...,n)$ are constants, in particular, if $c_1\neq 0$ and $c_2=0$, then $q(x,t)$ becomes

$$q(x,t)=-\lambda\pm\sqrt{4\mu-\lambda^2}-\sqrt{4\mu-\lambda^2}\,tg\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}(x-\sum_{j=1}^{n}\alpha_j t^j)\right). \tag{23}$$

The function (23) is the solution of the following loaded modified Burgers–KdV equation.

$$q_t+qq_x-q_{xx}+\left(\sum_{j=1}^{n} j\alpha_j t^{j-1}-\sqrt{4\mu-\lambda^2}\right)q_x=0.$$

When $(\lambda^2-4\mu)=0$,

$$q(\xi)=-\lambda-\frac{2c_2}{c_1+\xi c_2}, \tag{24}$$

where $\xi=x\pm\sqrt{\lambda^2-4\mu}t-\int_0^t\gamma(\tau)q(0,\tau)d\tau-\Omega^0$, c_1, c_2, and Ω^0 are arbitrary constants. The function $q(0,t)$ is found based on expression (24).

If $c_1=0$, $c_2\neq 0$, $(\lambda^2-4\mu)=0$ and $\gamma(t)$ is given as below

$$\gamma(t)=\frac{\sum_{j=1}^{n}\alpha_j t^j\sum_{j=1}^{n} j\alpha_j t^{j-1}}{-\lambda\sum_{j=1}^{n}\alpha_j t^j+2},$$

then $q(x,t)$ becomes

$$q(x,t)=-\lambda-\frac{2}{x-\sum_{j=1}^{n}\alpha_j t^j}. \tag{25}$$

We know that the function (25) satisfies the following loaded modified Burgers–KdV equation.

$$q_t+qq_x-q_{xx}+\left(\sum_{j=1}^{n} j\alpha_j t^{j-1}\right)q_x=0.$$

Graphical representation of the solutions of the loaded nonlinear equation

Above we have shown how to find the solutions of the loaded modified Burgers–KdV equation. In this section, we show the solutions of some loaded modified Burgers–KdV equations in 3D plot formats to make it easier to imagine.

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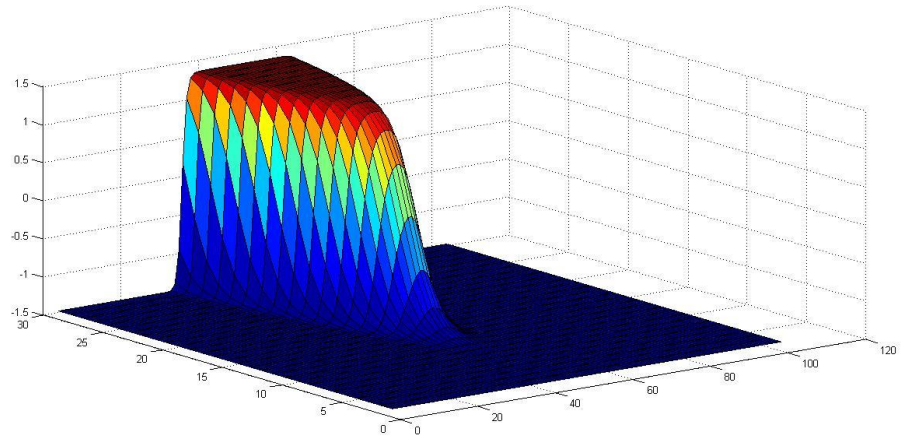

Fig. 1. The solution of the loaded modified Burgers–KdV equation for $\lambda = 2$, $\mu = -1$, $k = -1$ and $\gamma(t) = \frac{1}{2k}\left(2 + 8t - k^3(\lambda^2 - 4\mu)\right)$.

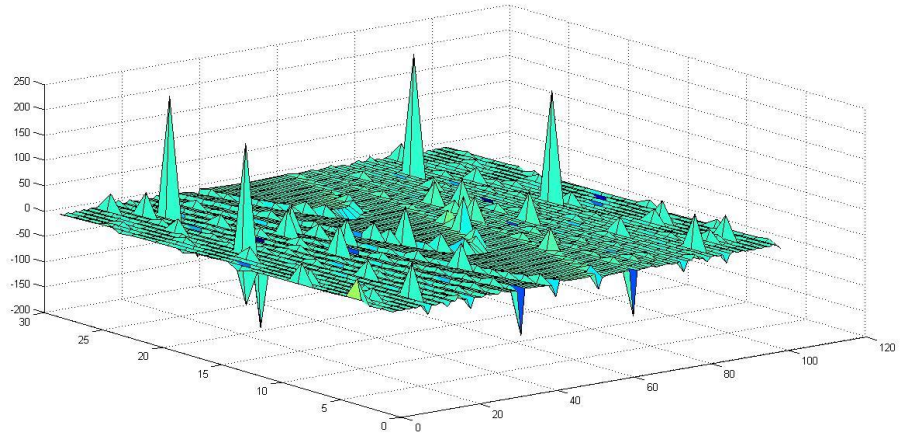

Fig. 2. The solution of the loaded modified Burgers–KdV equation for $\lambda = 3$, $\mu = 3$, $k = -1$ and $\gamma(t) = \frac{1}{2k}\left(2 + 8t + k^3(4\mu - \lambda^2)\right)$.

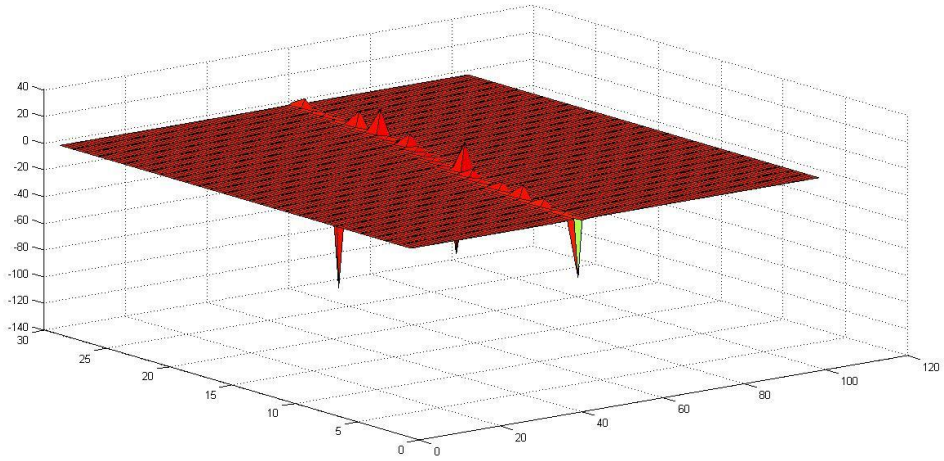

Fig. 3. The solution of the loaded modified Burgers–KdV equation for $\lambda = 2$, $\mu = 1$, $k = -1$ and $\gamma(t) = \frac{1}{k}\left(4 + 2t + 9t^2\right)$.

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